ISRAEL JOURNAL OF MATHEMATICS **159** (2007), 277–288
DOI: 10.1007/s11856-007-0047-1

EXTENSIONS OF HOLOMORPHIC MOTIONS

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ABSTRACT

We prove that a normalized holomorphic motion of a closed set E is induced by a holomorphic map into the Teichmüller space of E, denoted by $T(E)$, if and only if it can be extended to a normalized continuous motion of the Riemann sphere. We also prove that the extension can be chosen to have additional properties.

1. Basic definitions and the main theorem

Definition 1.1: Let V be a connected complex manifold with a basepoint x_0 and let E be a subset of the Riemann sphere $\widehat{\mathbb{C}}$. A **holomorphic motion of E over V** is a map $\phi\colon V \times E \to \widehat{\mathbb{C}}$ that has the following three properties:

(a) $\phi(x_0, z) = z$ for all z in E,

(b) the map $\phi(x, \cdot)\colon E \to \widehat{\mathbb{C}}$ is injective for each x in V, and

(c) the map $\phi(\cdot, z)\colon V \to \widehat{\mathbb{C}}$ is holomorphic for each z in E.

We will sometimes write $\phi(x, z)$ as $\phi_x(z)$ for x in V and z in E.

We say that V is the **parameter space** of the holomorphic motion ϕ.

We will always assume that ϕ is a **normalized** holomorphic motion; i.e. 0, 1, and ∞ belong to E and are fixed points of the map $\phi_x(\cdot)$ for every x in V.

* This research was partially supported by a PSC-CUNY grant.
Received December 21, 2005 and in revised form February 23, 2006

Definition 1.2: Let V and W be connected complex manifolds with basepoints, and f be a basepoint preserving holomorphic map of W into V. If ϕ is a holomorphic motion of E over V its **pullback** by f is the holomorphic motion

$$f^*(\phi)(x,z) = \phi(f(x),z) \quad \forall (x,z) \in W \times E \tag{1.1}$$

of E over W.

Throughout this paper we will assume that E is a closed subset of $\widehat{\mathbb{C}}$ and that $0,1,\infty \in E$. Associated to each such set E in $\widehat{\mathbb{C}}$, there is a contractible complex Banach manifold which we call the Teichmüller space of the closed set E, denoted by $T(E)$. This was first studied by G. Lieb in his doctoral dissertation [14] (see A. Epstein's dissertation [11] for a generalization). Furthermore, we can define a holomorphic motion

$$\Psi_E\colon T(E) \times E \to \widehat{\mathbb{C}}$$

of the closed set E over the parameter space $T(E)$. The precise definitions of $T(E)$ and Ψ_E and some of their properties are given in Sections 2 and 3.

In [15] it was shown that $T(E)$ is a universal parameter space for holomorphic motions of the closed set E over a simply connected complex Banach manifold. The space $T(E)$ and its various properties have been the subject of several papers in recent years; see [8], [9], [10], [15], and [16].

Definition 1.3: Let V be a path-connected Hausdorff space with a basepoint x_0. A normalized **continuous motion** of $\widehat{\mathbb{C}}$ over V is a continuous map $\phi\colon V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(i) $\phi(x_0, z) = z$ for all z in $\widehat{\mathbb{C}}$, and
(ii) for each x in V, the map $\phi(x,\cdot)$ is a homeomorphism of $\widehat{\mathbb{C}}$ onto itself that fixes the points 0, 1, and ∞.

As in Definition 1.1, we will sometimes write $\phi(x,\cdot)$ as $\phi_x(\cdot)$, and we will always assume that the continuous motion ϕ is normalized.

An important topic in the study of holomorphic motions, is the question of extensions. If E is a proper subset of $\widetilde{E}$ and $\phi\colon V \times E \to \widehat{\mathbb{C}}$, $\widetilde{\phi}\colon V \times \widetilde{E} \to \widehat{\mathbb{C}}$ are two maps, we say that $\widetilde{\phi}$ **extends** ϕ if $\widetilde{\phi}(x,z) = \phi(x,z)$ for all (x,z) in $V \times E$. If $\phi\colon V \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion, a natural question is whether there exists a holomorphic motion $\widetilde{\phi}\colon V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that extends ϕ. For $V = \Delta$ (the open unit disk), important results were obtained in [3] and in [21]. A complete affirmative answer was given in Slodkowski ([19]), where it was shown that any holomorphic motion of E over Δ can be extended to the whole sphere.

Slodkowski's theorem cannot be generalised to higher dimensional parameter spaces. This was shown by Hubbard with a two-dimensional Teichmüller space as a parameter space (see [5]). See also [7] and Appendix 2 in [10] for other interesting examples. The extension theorem of Bers and Royden in [3] was generalised in [5], [15], and [20].

In this paper we study the extension of holomorphic motions to continuous motions of $\widehat{\mathbb{C}}$.

THEOREM: *Let $\phi: V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion where V is a connected complex Banach manifold with a basepoint x_0. Then the following are equivalent:*

(i) *There is a continuous motion $\widetilde{\phi}: V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that extends ϕ.*

(ii) *There exists a basepoint preserving holomorphic map $F: V \to T(E)$ such that $F^*(\Psi_E) = \phi$.*

COROLLARY: *If the holomorphic motion ϕ can be extended to a continuous motion $\widetilde{\phi}$, then $\widetilde{\phi}$ can be chosen so that:*

(i) *the map $\widetilde{\phi}_x: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal for each x in V,*

(ii) *its Beltrami coefficient μ_x is a continuous function of x, and*

(iii) *for each x, the L^∞ norm of μ_x is bounded above by a number less than 1, that depends only on the Kobayashi distance from x to x_0, not on ϕ.*

Remark 1.4: The continuous motions $\widetilde{\phi}$ with properties (i) and (ii) are precisely the (normalized) quasiconformal motions of $\widehat{\mathbb{C}}$ defined by Sullivan and Thurston in ([21]) as we show in a forthcoming paper ([17]), where we also report some other properties of quasiconformal motions.

Remark 1.5: It was already known that if the complex manifold V is simply connected, then every holomorphic motion ϕ of E over V can be extended to a continuous motion $\widetilde{\phi}: V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. (See Theorem C in [15].)

Remark 1.6: Chirka introduces continuous motions in his study of extensions of holomorphic motions; see [4].

ACKNOWLEDGEMENT: I want to thank Clifford J. Earle for many discussions. He read an earlier draft and made some important comments. I also thank the participants of the Complex Analysis Seminar at the Graduate Center of the City University of New York and the Analysis seminar at Cornell University for their interesting questions. I am very grateful to the referee for several valuable and useful suggestions.

2. The Teichmüller space of E

2.1. Definition. Recall that a homeomorphism of $\widehat{\mathbb{C}}$ is called **normalized** if it fixes the points 0, 1, and ∞.

The normalized quasiconformal self-mappings f and g of $\widehat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The **Teichmüller space** $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$.

The basepoint of $T(E)$ is the E-equivalence class of the identity map.

2.2. $T(E)$ is a complex Banach manifold. Let $M(\mathbb{C})$ be the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\widehat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We define the quotient map

$$P_E \colon M(\mathbb{C}) \to T(E)$$

by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

In his doctoral dissertation ([14]), G. Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map P_E from $M(\mathbb{C})$ to $T(E)$ is a holomorphic split submersion. (This result is also proved in [10].)

2.3. The Teichmüller metric on $T(E)$ The Teichmüller distance $d_M(\mu,\nu)$ between μ and ν on $M(\mathbb{C})$ is defined by

$$d_M(\mu,\nu) = \tanh^{-1} \left\| \frac{\mu-\nu}{1-\bar{\mu}\nu} \right\|_\infty.$$

The **Teichmüller metric** on $T(E)$ is the quotient metric

$$d_{T(E)}(s,t) = \inf\{d_M(\mu,\nu) : \mu \text{ and } \nu \in M(\mathbb{C}), P_E(\mu) = s \text{ and } P_E(\nu) = t\}$$

for all s and t in $T(E)$. The Teichmüller metric on $T(E)$ is the same as its Kobayashi metric (see Proposition 7.30 in [10]).

2.4. Changing the basepoint. Let w be a normalized quasiconformal self-mapping of $\widehat{\mathbb{C}}$, and let $\widehat{E} = w(E)$. By definition, the **allowable map** g from $T(\widehat{E})$ to $T(E)$ maps the $\widehat{E}$-equivalence class of f (written as $[f]_{\widehat{E}}$) to the E-equivalence class of $f \circ w$ (written as $[f \circ w]_E$) for every normalized quasiconformal self-mapping f of $\widehat{\mathbb{C}}$.

Proposition 2.1: *The allowable map* $g: T(\widehat{E}) \to T(E)$ *is biholomorphic. If* μ *is the Beltrami coefficient of* w*, then* g *maps the basepoint of* $T(\widehat{E})$ *to the point* $P_E(\mu)$ *in* $T(E)$*.*

See Proposition 7.20 in [10] or Proposition 6.7 in [15]. The map g is also called the geometric isomorphism induced by the quasiconformal map w. (These are not the only biholomorphic maps between the spaces $T(E)$. The others are described in [9].)

2.5. Contractibility of $T(E)$. The following fact will be crucial in this paper.

Proposition 2.2: *There is a continuous basepoint preserving map* s *from* $T(E)$ *to* $M(\mathbb{C})$ *such that* $P_E \circ s$ *is the identity map on* $T(E)$*.*

For a complete proof we refer the reader to Proposition 7.22 in [10] (or Proposition 6.3 in [15]).

Since $M(\mathbb{C})$ is contractible, we conclude:

Corollary 2.3: *The space* $T(E)$ *is contractible.*

Remark 2.4: Here is an outline for the construction of $s(t)$ for t in $T(E)$. Choose an extremal μ in $M(\mathbb{C})$ such that $P_E(\mu) = t$. We set $s(t) = \mu$ in E. Let Ω be a connected component of $\widehat{\mathbb{C}} \setminus E$. On Ω, $s(t)$ is defined as follows. Choose a holomorphic universal cover $\pi: \Delta \to \Omega$ (where Δ is the open unit disk). Lift μ to Δ and let $\widetilde{\mu} = \pi^*(\mu)$ (the lift of μ). If $\pi(\zeta) = z$ we have

$$\widetilde{\mu}(\zeta) = \mu(z)\frac{\overline{\pi'(\zeta)}}{\pi'(\zeta)}.$$

Let $\widetilde{w}: \Delta \to \Delta$ be a quasiconformal map whose Beltrami coefficient is $\widetilde{\mu}$, and let $h: \partial\Delta \to \partial\Delta$ be the boundary homeomorphism. Let $w: \Delta \to \Delta$ be the barycentric extension of h and $\widetilde{\nu}$ be the Beltrami coefficient of w. Then, $\widetilde{\nu}$ is the lift of a uniquely determined L^∞ function ν on Ω. We set $s(t) = \nu$ in Ω. Then $\|\widetilde{\mu}\|_\infty = \|\mu|\Omega\|_\infty \leq k := \|\mu\|_\infty$; so,

$$\|s(t)|\Omega\|_\infty = \|\nu\|_\infty \leq c(k)$$

by Proposition 7 in [6], where $c(k)$ depends only on k and $0 \leq c(k) < 1$. Since Ω is any connected component of $\widehat{\mathbb{C}} \setminus E$, we conclude that $\|s(t)\|_\infty \leq \max(k, c(k))$.

3. Universal holomorphic motion of E

3.1. DEFINITION. The **universal holomorphic motion** Ψ_E of E over $T(E)$ is defined as follows:

$$\Psi_E(P_E(\mu), z) = w^{\mu}(z) \quad \text{for } \mu \in M(\mathbb{C}) \text{ and } z \in E.$$

The definition of P_E in §2.1 guarantees that Ψ_E is well-defined. It is a holomorphic motion since P_E is a holomorphic split submersion and $\mu \mapsto w^{\mu}(z)$ is a holomorphic map from $M(\mathbb{C})$ to $\widehat{\mathbb{C}}$ for every fixed z in $\widehat{\mathbb{C}}$ (by Theorem 11 in [1]). This holomorphic motion is "universal" in the following sense:

THEOREM 3.1: *Let ϕ: $V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. If V is simply connected, then there exists a unique basepoint preserving holomorphic map f: $V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof see Section 14 in [15].

3.2. AN EXTENSION OF Ψ_E. Let s: $T(E) \to M(\mathbb{C})$ be the continuous basepoint preserving section of the quotient map P_E described in Remark 2.4.

PROPOSITION 3.2: (i) *The map $\widetilde{\Psi}_E$: $T(E) \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by the formula*

$$\widetilde{\Psi}_E(t, z) = w^{s(t)}(z), \quad (t, z) \in T(E) \times \widehat{\mathbb{C}},$$

is a continuous motion that extends the universal holomorphic motion

$$\Psi_E \colon T(E) \times E \to \widehat{\mathbb{C}}.$$

(ii) *For t in $T(E)$, $\|s(t)\|_\infty$ is bounded above by a number between 0 and 1, that depends only on $d_{T(E)}(0, t)$.*

Proof: (i) Properties (i) and (ii) of Definition 1.3 are obviously satisfied by the map $\widetilde{\Psi}_E$. The continuity of $\widetilde{\Psi}_E$ follows from Lemma 17 of [1], which says that $w^{\mu_n} \to w^{\mu}$ uniformly in the spherical metric if $\mu_n \to \mu$ in $M(\mathbb{C})$. Therefore, $\widetilde{\Psi}_E$: $T(E) \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a normalized continuous motion.

Finally, we have

$$\Psi_E(t, z) = \Psi_E(P_E(s(t)), z) = w^{s(t)}(z) = \widetilde{\Psi}_E(t, z)$$

for all $(t, z) \in T(E) \times E$. Therefore, $\widetilde{\Psi}_E$ extends Ψ_E.

(ii) Given t in $T(E)$, choose an extremal μ in $M(\mathbb{C})$ so that $P_E(\mu) = t$. Then

$$d_{T(E)}(0, t) = \frac{1}{2}\log K \quad \text{where } K = \frac{1+k}{1-k} \text{ and } k = \|\mu\|_\infty.$$

By Remark 2.4, $\|s(t)\|_\infty \leq \max(c(k), k)$. ∎

4. Two lemmas

The first lemma was proved in [15], where it is Lemma 12.1. Let B be a path-connected topological space and $\mathcal{H}(\widehat{\mathbb{C}})$ be the group of homeomorphisms of $\widehat{\mathbb{C}}$ onto itself, with the topology of uniform convergence in the spherical metric. This topology makes $\mathcal{H}(\widehat{\mathbb{C}})$ a topological group (see [2]). The symbol E has its usual meaning.

Lemma 4.1: *Let $h\colon B \to \mathcal{H}(\widehat{\mathbb{C}})$ be a continuous map such that $h(t)(e) = e$ for all t in B and for all e in E. If $h(t_0)$ is isotopic to the identity rel E for some fixed t_0 in B, then $h(t)$ is isotopic to the identity rel E for all t in B.*

Lemma 4.2: *Let $s\colon T(E) \to M(\mathbb{C})$ satisfy the conditions of Proposition 2.2, and let $\psi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be any homeomorphism. There is at most one point t in $T(E)$ such that ψ is isotopic to $w^{s(t)}$ rel E.*

Proof: If $w^{s(t)}$ and $w^{s(t')}$ are both isotopic to ψ rel E, then they are E-equivalent, so $t = P_E(s(t)) = P_E(s(t')) = t'$. ∎

5. Proof of the main theorem

Let $\phi\colon V \times E \to \widehat{\mathbb{C}}$ be the given holomorphic motion, and let $s\colon T(E) \to M(\mathbb{C})$ satisfy the conditions of Proposition 2.2.

Part 1: (ii) implies (i): Define $\widetilde{F}\colon V \to M(\mathbb{C})$ by $\widetilde{F} = s \circ F$. Then $\widetilde{F}\colon V \to M(\mathbb{C})$ is a basepoint preserving continuous map. Define $\widetilde{\phi}\colon V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$\widetilde{\phi}(x, z) = w^{\widetilde{F}(x)}(z)$$

for all x in V and for all z in $\widehat{\mathbb{C}}$. Clearly, $\widetilde{\phi}(x_0, z) = z$ for all z in $\widehat{\mathbb{C}}$. The continuity of $\widetilde{\phi}$ is similar to the continuity of $\widetilde{\Psi}_E$ in the proof of Proposition 3.2(i). So, $\widetilde{\phi}\colon V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a continuous motion.

Finally, for all z in E, we have $\phi(x, z) = F^*(\Psi_E)(x, z) = \Psi_E(F(x), z) = \Psi_E(P_E(s(F(x))), z) = w^{s(F(x))}(z) = w^{\widetilde{F}(x)}(z) = \widetilde{\phi}(x, z)$. Hence $\widetilde{\phi}$ extends ϕ.

Part 2: (i) implies (ii): Let $\widetilde{\phi}\colon V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous motion that extends ϕ. Let S be the set of points x in V with the following property: there exists a neighborhood N of x and a holomorphic map $h\colon N \to T(E)$ such that $w^{s(h(x'))}$ is isotopic to $\widetilde{\phi}_{x'}$ rel E for all x' in N. We claim that $S = V$.

It is clear that S is an open set. To see that it contains the basepoint x_0 of V, choose a simply connected neighborhood N of x_0 in V, and give N the basepoint x_0. By Theorem 3.1, there exists a basepoint preserving holomorphic map h: $N \to T(E)$ such that $h^*(\Psi_E) = \phi$ on $N \times E$. Define

$$H(x) = (w^{s(h(x))})^{-1} \circ \widetilde{\phi}_x$$

for each x in N. Clearly, $H(x_0)$ is the identity. Also, for all x in N, and for all z in E,

$$\widetilde{\phi}_x(z) = \widetilde{\phi}(x, z) = \phi(x, z) = \Psi_E(h(x), z) = w^{s(h(x))}(z).$$

Hence, for all z in E, $H(x)(z) = z$. Since $H(x)$ is continuous in x, it follows from Lemma 4.1 that $H(x)$ is isotopic to the identity rel E. Hence, for each x in N, $w^{s(h(x))}$ is isotopic to $\widetilde{\phi}_x$ rel E. This shows that x_0 belongs to S.

Now we shall prove that S is closed. Let y belong to the closure of S, choose a simply connected neighborhood B of y, and give B a basepoint p in S. Let

$$\widehat{E} = \phi_p(E) = \{\phi(p, z) : z \in E\}$$

and consider

$$\widehat{\phi}(x, \phi_p(z)) = \phi(x, z) \quad \forall (x, z) \in B \times E.$$

This is a holomorphic motion of $\widehat{E}$ over B with basepoint p. By Theorem 3.1, there exists a basepoint preserving holomorphic map f: $B \to T(\widehat{E})$ such that $f^*(\Psi_{\widehat{E}}) = \widehat{\phi}$ on $B \times \widehat{E}$ (where $\Psi_{\widehat{E}}$: $T(\widehat{E}) \times \widehat{E} \to \widehat{\mathbb{C}}$ is the universal holomorphic motion of $\widehat{E}$). This means

$$\Psi_{\widehat{E}}(f(x), \phi_p(z)) = \widehat{\phi}(x, \phi_p(z)) \tag{5.1}$$

for all x in B and for all z in E.

Since $p \in S$, there is a point t in $T(E)$ such that $\widetilde{\phi}_p$ is isotopic to $w^{s(t)}$ rel E. Thus, $w^{s(t)}$ maps E onto $\widehat{E}$; so it induces a biholomorphic map g: $T(\widehat{E}) \to T(E)$ as in §2.4. Define $\widehat{h}$: $B \to T(E)$ by $\widehat{h} = g \circ f$. We are going to prove that $w^{s(\widehat{h}(x))}$ is isotopic to $\widetilde{\phi}_x$ rel E for all x in B.

Note that f maps p to the basepoint of $T(\widehat{E})$ and by Proposition 2.1, g maps $f(p)$ to the point $P_E(s(t))$ in $T(E)$. Therefore, $\widehat{h}(p) = P_E(s(t))$ and since $\widehat{h}(p) = P_E(s(\widehat{h}(p)))$, we have $P_E(s(t)) = P_E(s(\widehat{h}(p)))$. That means, $w^{s(t)}$ is isotopic to $w^{s(\widehat{h}(p))}$ rel E; so $\widetilde{\phi}_p$ is isotopic to $w^{s(\widehat{h}(p))}$ rel E.

Let

$$\widehat{H}(x) = (w^{s(\widehat{h}(x))})^{-1} \circ \widetilde{\phi}_x \tag{5.2}$$

for all x in B. By the above discussion, $\widehat{H}(p)$ is isotopic to the identity rel E.

We have the standard projection map

$$P_{\widehat{E}}\colon M(\mathbb{C}) \to T(\widehat{E}),$$

and $\widehat{s}\colon T(\widehat{E}) \to M(\mathbb{C})$ is a continuous basepoint preserving map such that $P_{\widehat{E}} \circ \widehat{s}$ is the identity map on $T(\widehat{E})$. Since $\widetilde{\phi}_p$ is isotopic to $w^{s(t)}$ rel E, and $\widetilde{\phi}_p(z) = \phi_p(z)$ for all z in E, it follows that

$$\phi_p(z) = w^{s(t)}(z) \tag{5.3}$$

for all z in E. Furthermore, for all $x \in B$, and $z \in E$, we have:

$$\widetilde{\phi}_x(z) = \phi_x(z) = \widehat{\phi}_x(\phi_p(z)) = \Psi_{\widehat{E}}(f(x), \phi_p(z))$$

by Equation 5.1. And $\Psi_{\widehat{E}}(f(x), \phi_p(z)) = w^{\widehat{s}(f(x))}(\phi_p(z)) = w^{\widehat{s}(f(x))}(w^{s(t)}(z))$ by Equation 5.3. We conclude

$$\widetilde{\phi}_x(z) = w^{\widehat{s}(f(x))}(w^{s(t)}(z)) \tag{5.4}$$

for all x in B, and for all z in E.

For all x in B, we have $\widehat{h}(x) = g(f(x))$. Also, $f(x) = P_{\widehat{E}}(\widehat{s}(f(x))) = [w^{\widehat{s}(f(x))}]_{\widehat{E}}$ and by §2.4,

$$g\colon [w^{\widehat{s}(f(x))}]_{\widehat{E}} \mapsto [w^{\widehat{s}(f(x))} \circ w^{s(t)}]_E.$$

Therefore,

$$\widehat{h}(x) = [w^{\widehat{s}(f(x))} \circ w^{s(t)}]_E.$$

We also have $\widehat{h}(x) = P_E(s(\widehat{h}(x))) = [w^{s(\widehat{h}(x))}]_E$ for all x in B. Hence, for all x in B, and for all z in E, we have

$$w^{\widehat{s}(f(x))}(w^{s(t)}(z)) = w^{s(\widehat{h}(x))}(z). \tag{5.5}$$

Therefore, by Equations 5.4 and 5.5, we get $\widetilde{\phi}_x(z) = w^{s(\widehat{h}(x))}(z)$ for all x in B and for all z in E. Hence, by Equation 5.2, $\widehat{H}(x)(z) = z$ for all x in B, and for all z in E. Since $\widehat{H}$ is continuous in x, it follows from Lemma 4.1 that $\widehat{H}(x)$ is isotopic to the identity rel E for all x in B. Therefore $w^{s(\widehat{h}(x))}$ is isotopic to $\widetilde{\phi}_x$ rel E for all x in B. Hence B is contained in S. In particular, $y \in S$, so S is closed. As S is also open and nonempty, $S = V$.

We now define a holomorphic map $F\colon V \to T(E)$ as follows. Given any x in V, choose a neighborhood N of x and a holomorphic map $h\colon N \to T(E)$ such that $w^{s(h(x'))}$ is isotopic to $\widetilde{\phi}_{x'}$ rel E for all x' in N. Set $F = h$ in N. Lemma 4.2

implies that F is well-defined on all of V. It is obviously holomorphic, and $w^{s(F(x))}$ is isotopic to $\widetilde{\phi}_x$ rel E for all x in V.

Finally, for all x in V, and for all z in E, we have

$$F^*(\Psi_E)(x,z) = \Psi_E(F(x),z) = \Psi_E(P_E(s(F(x))),z) = w^{s(F(x))}(z)$$

and $\phi(x,z) = \widetilde{\phi}(x,z) = \widetilde{\phi}_x(z) = w^{s(F(x))}(z)$ (since $w^{s(F(x))}$ is isotopic to $\widetilde{\phi}_x$ rel E for all x in V). Therefore $F^*(\Psi_E)(x,z) = \phi(x,z)$ for all x in V and for all z in E. This completes the proof.

Remark 5.1: If F and G are two basepoint preserving holomorphic maps from V into $T(E)$ such that $F^*(\Psi_E) = G^*(\Psi_E) = \phi$, then it follows from Lemma 12.2 in [15] that $F = G$. Thus, if a basepoint preserving holomorphic map $F\colon V \to T(E)$ such that $F^*(\Psi_E) = \phi$ exists, then it is unique.

6. Proof of the corollary

If ϕ can be extended to a continuous motion of $\widehat{\mathbb{C}}$, then by our main theorem there is a basepoint preserving holomorphic map $F\colon V \to T(E)$ such that $F^*(\Psi_E) = \phi$.

Using the continuous map $s\colon T(E) \to M(\mathbb{C})$ described in Remark 2.4, define the continuous motion $\widetilde{\phi}\colon V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as in Part 1 of the proof of the main theorem. We showed there that $\widetilde{\phi}$ extends ϕ, and it clearly satisfies conditions (i) and (ii) of the Corollary.

For (iii), let x be in V $(x \neq x_0)$, and let $F\colon V \to T(E)$ be the holomorphic map above. Since the Teichmüller metric on $T(E)$ is the same as its Kobayashi metric (see §2.3), we have $d_{T(E)}(0,t) \leq \rho_V(x_0,x)$ where $F(x) = t$ and 0 denotes the basepoint in $T(E)$. Choose an extremal μ in $M(\mathbb{C})$ such that $P_E(\mu) = F(x)$. This means that $d_{T(E)}(0, P_E(\mu)) = d_M(0_M, \mu)$ where 0_M denotes the basepoint in $M(\mathbb{C})$. We have

$$d_{T(E)}(F(x_0),F(x)) = \frac{1}{2}\log\frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty} \leq \rho_V(x_0,x)$$

which gives

$$\|\mu\|_\infty \leq \frac{\exp(2\rho_V(x_0,x))-1}{\exp(2\rho_V(x_0,x))+1} < 1.$$

Since $\widetilde{\phi}(x,z) = w^{\widetilde{F}(x)}(z)$, where $\widetilde{F} = s \circ F$, it follows from Part (ii) of Proposition 3.2, that $\|w^{\widetilde{F}(x)}\|_\infty$ is bounded above by a number between 0 and 1, that depends only on $\rho_V(x_0,x)$.

7. An example

Remark 7.1: If $\phi: V \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion where V is a simply connected complex Banach manifold, it follows from Theorem 3.1, and the main theorem of this paper, that there always exists a normalized continuous motion $\widetilde{\phi}: V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that extends ϕ. Furthermore, $\widetilde{\phi}$ has the properties (i), (ii) and (iii) of the Corollary.

As already pointed out in Chirka ([4]), there are simple examples of holomorphic motions that cannot be extended to continuous motions of $\widehat{\mathbb{C}}$. I am grateful to Clifford Earle for the following explicit example.

Let $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ and choose some basepoint a in Δ^*. Let $E := \{0, 1, a, \infty\}$.

Proposition 7.2: *Set $\phi(t, z) = z$ for all (t, z) in $\Delta^* \times \{0, 1, \infty\}$ and $\phi(t, a) = t$ for all t in Δ^*. Then ϕ is a holomorphic motion of E over Δ^* that cannot be extended to a continuous motion of $\widehat{\mathbb{C}}$ over Δ^*.*

Proof: We follow Chirka's argument. Suppose $\widetilde{\phi}$ is such an extension. For each ζ in $\mathbb{C} \setminus \{0\}$, let $\gamma_\zeta: [0, 2\pi] \to \mathbb{C} \setminus \{0\}$ be the closed curve

$$\gamma_\zeta(\theta) = \widetilde{\phi}(ae^{i\theta}, \zeta)$$

for θ in $[0, 2\pi]$.

Since $\widetilde{\phi}$ is a continuous motion, the winding number of γ_ζ about zero is a continuous function of ζ. But that winding number is zero when $\zeta = 1$ and one when $\zeta = a$. ∎

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